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**FRACTIONAL DIFFERENTIAL EQUATIONS:
 α -ENTIRE SOLUTIONS,
REGULAR AND IRREGULAR SINGULARITIES**

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Abstract

We consider fractional differential equations of order $\alpha \in (0, 1)$ for functions of one independent variable $t \in (0, \infty)$ with the Riemann-Liouville and Caputo-Dzhrbashyan fractional derivatives. A precise estimate for the order of growth of α -entire solutions is given. An analog of the Frobenius method for systems with regular singularity is developed. For a model example of an equation with a kind of an irregular singularity, a series for a formal solution is shown to be convergent for $t > 0$ (if α is an irrational number poorly approximated by rational ones) but divergent in the distribution sense.

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Key Words and Phrases: fractional differential equation; Riemann-Liouville derivative; Caputo-Dzhrbashyan derivative; regular singularity; irregular singularity

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1. Introduction

Fractional differential equations are widely used for modeling anomalous relaxation and diffusion phenomena; see [3, 12] for further references. Meanwhile the mathematical theory of such equations is still in its initial stage. In particular, a systematic development of the analytic theory of fractional differential equations with variable coefficients was initiated only recently, in the paper by Kilbas, Rivero, Rodríguez-Germá, and Trujillo [11] (see also Section 7.5 in [12]). For equations of order $\alpha \in (0, 1)$, of the form

$$(D_{0+}^{\alpha} u)(t) = a(t)u(t), \quad t > 0, \quad (1)$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative, or

$$(\mathbb{D}^{(\alpha)} u)(t) = a(t)u(t), \quad t > 0, \quad (2)$$

where $\mathbb{D}^{(\alpha)}$ is the Caputo-Dzhrbashyan fractional derivative, their main results are as follows. If $a(t) = A(t^{\alpha})$, and $A(z)$ is a real function possessing an absolutely convergent Taylor expansion on an interval $|z| < \theta$, then the equation (1) possesses a solution of the form

$$u(t) = t^{\alpha-1} \sum_{n=0}^{\infty} a_n t^{\alpha n}, \quad 0 < t < \theta,$$

while the equation (2) has a solution

$$u(t) = \sum_{n=0}^{\infty} b_n t^{\alpha n}, \quad 0 \leq t < \theta.$$

In both cases the solutions are unique, if appropriate initial conditions are prescribed.

Thus, for example, the property of α -analyticity of the coefficient $a(t)$ (defined above) implies a similar property of a solution of the equation (2). In fact, we have $u(t) = U(t^{\alpha})$, where U is holomorphic in a disk $\{z \in \mathbb{C}, |z| < \theta\}$. The coefficient a may be complex-valued as well.

The above results open the way for developing a theory of α -analytic solutions of fractional differential equations in the spirit of classical analytic theory of ordinary differential equations. Here we give some results in this direction.

If in (2) $a(t) = A(t^\alpha)$ where A is an entire function, then the above results from [11] with $\theta = \infty$ (stated there in a weaker form, only for real arguments of analytic functions, than actually proved) show that the solution of the Cauchy problem for the equation (2) is of the form $u(t) = U(t^\alpha)$, where U is an entire function. Following [11, 12] we call such solutions α -entire. In particular, that is true, if A is a polynomial. A natural question is about the order of U (here we investigate this subject just for the equation (2) since its properties are closer to those of ordinary differential equations). It is known (see, for example, [2] or [8]) that every nontrivial solution of the equation $u^{(k)}(z) = A(z)u(z)$, $k \in \mathbb{N}$, with a polynomial coefficient A , is an entire function of order $1 + \deg(A)/k$. In this paper we prove that the orders of the entire functions U corresponding to solutions of (2) do not exceed $(1 + \deg(A))/\alpha$. As $\alpha \rightarrow 1$, this agrees with the above differential equation result. On the other hand, if $\deg(A) = 0$, that is $A(z) = \lambda$, $\lambda \in \mathbb{C}$, then $U(z) = E_\alpha(\lambda z)$, where E_α is the Mittag-Leffler function whose order is $1/\alpha$ [3, 12], which shows the exactness of our general estimate.

Next, we investigate systems of fractional equations with regular singularity, that is the equations

$$t^\alpha (D_{0+}^\alpha u)(t) = A(t^\alpha)u(t) \quad (3)$$

and

$$t^\alpha (\mathbb{D}^{(\alpha)} u)(t) = A(t^\alpha)u(t), \quad (4)$$

where $A(z)$ is a holomorphic matrix-function. Under some assumptions, we prove that formal power series solutions of (3) and (4) converge near the origin and develop an analog of the classical Frobenius method of finding a solution. For scalar equations, the latter problem was considered in [12, 15].

Finally, in order to clarify characteristic features of fractional equations with irregular singularity, we study a model example, the equation

$$t^{2\alpha} (\mathbb{D}^{(\alpha)} u)(t) = \lambda u(t), \quad \lambda \in \mathbb{C}, \quad (5)$$

where, as before, $0 < \alpha < 1$. Assuming that α is irrational and satisfies a Diophantine condition (that is α is poorly approximated by rational numbers), we construct a kind of a formal solution of (5) convergent for $t > 0$. We prove that the series for the formal solution does not converge in the distribution sense, within a theory of distributions associated with the fractional calculus (see [16, 18]). Thus, the formal solution $u(t)$ cannot be interpreted as a distribution solution. It is interesting that $u(t)$ is

closely connected with a class of analytic functions with irregular behavior introduced by Hardy [6].

2. Preliminaries

2.1. Fractional derivatives and integrals [3, 12, 18]. Let $\alpha \in (0, 1)$ be a fixed number. The Riemann-Liouville fractional integral of order α of a function $\varphi \in L_1(0, T)$ is defined as

$$(I_{0+}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \varphi(\tau) d\tau, \quad 0 < t \leq T.$$

The Riemann-Liouville fractional derivative of order α is given by the expression

$$(D_{0+}^\alpha \varphi)(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} \varphi(\tau) d\tau, \quad 0 < t \leq T,$$

that is $(D_{0+}^\alpha \varphi)(t) = \frac{d}{dt} (I_{0+}^{1-\alpha} \varphi)(t)$, provided the fractional integral $I_{0+}^{1-\alpha} \varphi$ is an absolutely continuous function. If φ is defined on the whole half-axis $(0, \infty)$, then $I_{0+}^\alpha \varphi$ and $D_{0+}^\alpha \varphi$ are also defined on $(0, \infty)$. Below we will consider just this case.

Note that the Riemann-Liouville derivative is defined for some functions with a singularity at the origin. For example, if $\varphi(t) = t^d$, $d > -1$, then

$$(D_{0+}^\alpha \varphi)(t) = \frac{\Gamma(d+1)}{\Gamma(d+1-\alpha)} t^{d-\alpha}, \quad (6)$$

so that $D_{0+}^\alpha \varphi = 0$, if $\varphi(t) = t^{\alpha-1}$. For $\varphi(t) = t^d$, $d > -1$, we have also

$$(I_{0+}^\alpha \varphi)(t) = \frac{\Gamma(d+1)}{\Gamma(d+1+\alpha)} t^{d+\alpha}. \quad (7)$$

The Riemann-Liouville fractional differentiation and integration are inverse to each other in the following sense. If $\varphi \in L_1(0, T)$, then $D_{0+}^\alpha I_{0+}^\alpha \varphi = \varphi$. The equality $I_{0+}^\alpha D_{0+}^\alpha \varphi = \varphi$ holds under the stronger assumption that $\varphi = I_{0+}^\alpha \psi$ with some $\psi \in L_1(0, T)$. The latter is equivalent to the conditions of absolute continuity of $I_{0+}^{1-\alpha} \varphi$ on $[0, T]$ and the equality $(I_{0+}^{1-\alpha} \varphi)(0) = 0$.

Let a function φ be continuous on $[0, T]$ and possess the Riemann-Liouville fractional derivative of order α . The function

$$\begin{aligned} \left(\mathbb{D}^{(\alpha)} \varphi \right) (t) &= \frac{1}{\Gamma(1-\alpha)} \left[\frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} \varphi(\tau) d\tau - t^{-\alpha} \varphi(0) \right] \\ &= \left(D_{0+}^{\alpha} \varphi \right) (t) - \frac{1}{t^{\alpha} \Gamma(1-\alpha)} \varphi(0) \end{aligned} \quad (8)$$

is called *the Caputo-Dzhrbashyan, or regularized, fractional derivative*. If φ is absolutely continuous on $[0, T]$, then

$$\left(\mathbb{D}^{(\alpha)} \varphi \right) (t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \varphi'(\tau) d\tau. \quad (9)$$

In contrast to D_{0+}^{α} , $\mathbb{D}^{(\alpha)}$ is defined only on continuous functions and vanishes on constant functions. In most of physical applications, equations with $\mathbb{D}^{(\alpha)}$ are used, because a solution of an equation with the Riemann-Liouville derivative typically has a singularity at the origin $t = 0$, so that the initial state of a system to be described by the equation is not defined.

Let $v(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function. Consider the Caputo-Dzhrbashyan derivative of the function

$$\varphi(t) = v(t^{\alpha}) = \sum_{n=0}^{\infty} c_n t^{\alpha n}.$$

It follows from (6) and (8) that

$$\left(\mathbb{D}^{(\alpha)} \varphi \right) (t) = \sum_{n=1}^{\infty} c_n \beta(n) t^{\alpha(n-1)} = (\mathfrak{D}_{\alpha} v) (t^{\alpha})$$

where

$$\beta(n) = \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1 - \alpha)}, \quad (10)$$

and the operator

$$(\mathfrak{D}_{\alpha} v) (z) = \sum_{n=1}^{\infty} c_n \beta(n) z^{n-1}$$

is known as the Gelfond-Leontiev (G-L) operator of generalized differentiation (see [13, pp. 72-85], [18, pp. 426-427]; in fact, \mathfrak{D}_{α} is defined for wider classes of functions). As a matter of fact, this is a generalized G-L

differentiation operator with respect to the Mittag-Leffler function (see e.g. in [13, 18]):

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad \alpha > 0,$$

and for the G-L operators with respect to (arbitrary) entire function $\varphi(z)$, see the original work [4].

The operator, right inverse to \mathfrak{D}_α , has the form

$$(\mathfrak{I}_\alpha f)(z) = \frac{z}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(zt^\alpha) dt$$

(see Sect. 22.3 in [18]). It will be convenient to make a change of variables setting $z = Re^{i\theta}$, $r = Rt^\alpha$. Then

$$(\mathfrak{I}_\alpha f)(Re^{i\theta}) = \frac{e^{i\theta}}{\alpha\Gamma(\alpha)} \int_0^R \left[\left(\frac{R}{r} \right)^{1/\alpha} - 1 \right]^{\alpha-1} f(re^{i\theta}) dr. \quad (11)$$

If $f(z) = \sum_{k=0}^{\infty} f_k z^k$, then

$$(\mathfrak{I}_\alpha f)(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k + 1 + \alpha)} f_k z^{k+1},$$

and it is easy to check that

$$(\mathfrak{I}_\alpha \mathfrak{D}_\alpha v)(z) = v(z) - v(0). \quad (12)$$

2.2. A class of distributions. Spaces of test functions and distributions behaving reasonably under the action of fractional integration operators were introduced by Rubin [16] (for a brief exposition see also [18]; both in [18] and [16] there are references regarding other approaches and earlier publications in this field). Proceeding from [16], it is easy to come to a class of distributions, where the Caputo-Dzhrbashyan derivative $\mathbb{D}^{(\alpha)}$ is defined in a natural way.

Let $\mathcal{S}(0, \infty)$ be the Schwartz space of smooth functions on $[0, \infty)$ with rapid decay at infinity. Denote

$$\mathcal{S}_+ = \{\varphi \in \mathcal{S}(0, \infty) : \varphi^{(l)}(0) = 0, \quad l = 0, 1, 2, \dots\},$$

$$\Phi_+^{1-\alpha} = \left\{ \varphi \in \mathcal{S}_+ : \int_0^\infty \varphi(x) x^{1-\alpha-k} dx = 0, \quad k = 1, 2, \dots \right\},$$

$$\Phi_+^{\alpha-1} = \left\{ \varphi \in \mathcal{S}_+ : \int_0^\infty \varphi(x) x^{\alpha-1-k} dx = 0, \quad k = 0, 1, 2, \dots \right\}.$$

These spaces are interpreted as topological vector spaces with the topologies induced from $\mathcal{S}(0, \infty)$; see [16] for various descriptions of these topologies including the description by seminorms.

Together with the Riemann-Liouville fractional integration operator I_{0+}^α , it is convenient to use the operator

$$(I_-^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (\tau - t)^{\alpha-1} \varphi(\tau) d\tau, \quad t > 0.$$

If φ, ψ are sufficiently good functions, for example, if $\varphi \in L_p(0, \infty)$, $\psi \in L_q(0, \infty)$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 2 - \alpha$, then

$$\int_0^\infty \varphi(x) (I_{0+}^{1-\alpha} \psi)(x) dx = \int_0^\infty \psi(x) (I_-^{1-\alpha} \varphi)(x) dx$$

(see Section 2.5.1 in [18]). We will write this in the notation

$$\langle \varphi, I_{0+}^{1-\alpha} \psi \rangle = \langle \psi, I_-^{1-\alpha} \varphi \rangle.$$

In particular, if $\psi = u'$, where $u \in \mathcal{S}(0, \infty)$, then

$$\langle \varphi, \mathbb{D}^{(\alpha)} u \rangle = \langle u', I_-^{1-\alpha} \varphi \rangle. \quad (13)$$

It is known [16] that $I_-^{1-\alpha}$ acts continuously from $\Phi_+^{1-\alpha}$ onto $\Phi_+^{\alpha-1}$. Therefore the identity (13) can be used to define $\mathbb{D}^{(\alpha)} u$ as a distribution from $(\Phi_+^{1-\alpha})'$, if $u \in C^1(0, \infty)$, and u' has no more than a power-like growth near zero and infinity. This definition agrees with the classical one: if u is continuously differentiable at the origin too, then

$$\begin{aligned} \langle u', I_-^{1-\alpha} \varphi \rangle &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty u'(x) dx \int_x^\infty (t-x)^{-\alpha} \varphi(t) dt \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \varphi(t) dt \int_0^t (t-x)^{-\alpha} u'(x) dx = \langle \varphi, \mathbb{D}^{(\alpha)} u \rangle, \end{aligned}$$

where $\mathbb{D}^{(\alpha)} u$ is understood in the sense of (9).

A typical example of a function from $\Phi_+^{1-\alpha}$ is the function

$$\varkappa_\alpha(x) = x^{\alpha-2} \exp\left(-\frac{\log^2 x}{4}\right) \sin\left(\frac{\pi}{2} \log x\right).$$

It is clear that $\varkappa_\alpha \in \mathcal{S}_+$. Next, if

$$\varkappa(x) = \exp\left(-\frac{\log^2 x}{4}\right) \sin\left(\frac{\pi}{2} \log x\right),$$

then we can write explicitly the Mellin transform

$$\tilde{\varkappa}(z) = \int_0^\infty x^{z-1} \varkappa(x) dx.$$

Namely, by the formula (4.133.1) from [5],

$$\tilde{\varkappa}(z) = 2 \int_0^\infty e^{-t^2/4} \sinh(z t) \sin \frac{\pi t}{2} dt = 2\sqrt{\pi} e^{z^2 - \frac{\pi^2}{4}} \sin \pi z. \quad (14)$$

In particular,

$$\int_0^\infty \varkappa_\alpha(x) x^{1-\alpha-k} dx = \tilde{\varkappa}(-k) = 0, \quad k = 0, 1, 2, \dots,$$

so that indeed $\varkappa_\alpha \in \Phi_+^{1-\alpha}$.

In order to have a full concept of a class of distributions, one needs a result regarding density of a space of test functions in some space of integrable functions. This gives a one-to-one correspondence between ordinary functions and distributions they generate. Here we present such a result though it will not be used directly in this paper. For similar properties in other situations see [17].

PROPOSITION 1. *The space $\Phi_+^{1-\alpha}$ is dense in $L_p((0, \infty), t^{-1} dt)$, $1 \leq p < \infty$.*

P r o o f. Let $R = \int_0^\infty \varkappa_\alpha(t) t^{-1} dt$. We have

$$R = \tilde{\varkappa}(\alpha - 2) = 2\sqrt{\pi} \exp\left((\alpha - 2)^2 - \frac{\pi^2}{4}\right) \sin \pi \alpha > 0.$$

Denote

$$z_N(x) = \frac{N}{R} \kappa_\alpha(x^N), \quad N = 1, 2, \dots$$

Then

$$\int_0^\infty z_N(x) x^{-1} dx = \frac{1}{R} \int_0^\infty \kappa_\alpha(t) t^{-1} dt = 1. \quad (15)$$

Suppose that $f \in L_p((0, \infty), t^{-1} dt)$. Consider the so-called Mellin convolution

$$(z_N *_M f)(t) = \int_0^\infty z_N(\tau) f\left(\frac{t}{\tau}\right) \tau^{-1} d\tau,$$

that is actually the convolution on the multiplicative group $(0, \infty)$ (note that $t^{-1} dt$ is a Haar measure on that group). Obviously, $z_N *_M f \in \Phi_+^{1-\alpha}$. Denote by $\|\cdot\|_p$ the norm in $L_p((0, \infty), t^{-1} dt)$.

Using (15) we can write

$$(z_N *_M f)(t) - f(t) = \int_0^\infty z_N(\tau) \left[f\left(\frac{t}{\tau}\right) - f(t) \right] \tau^{-1} d\tau.$$

By the generalized Minkowski inequality,

$$\|z_N *_M f - f\|_p \leq \int_0^\infty |z_N(\tau)| \left\{ \int_0^\infty \left| f\left(\frac{t}{\tau}\right) - f(t) \right|^p t^{-1} dt \right\}^{1/p} d\tau.$$

Next we use the L_p -continuity of shifts on the multiplicative group $(0, \infty)$ (see [10], Theorem 20.4). For any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\left\{ \int_0^\infty \left| f\left(\frac{t}{\tau}\right) - f(t) \right|^p t^{-1} dt \right\}^{1/p} < \varepsilon,$$

if $|\tau - 1| < \delta$. Thus,

$$\|z_N *_M f - f\|_p \leq 2\|f\|_p \int_{|\tau-1| \geq \delta} |z_N(\tau)| \tau^{-1} d\tau + \varepsilon \int_{|\tau-1| < \delta} |z_N(\tau)| \tau^{-1} d\tau. \quad (16)$$

Note that

$$\int_0^\infty |z_N(\tau)| \tau^{-1} d\tau = \frac{N}{R} \int_0^\infty |\kappa_\alpha(x^N)| x^{-1} dx = \frac{1}{R} \int_0^\infty |\kappa_\alpha(t)| t^{-1} dt = C_1$$

where the constant C_1 does not depend on N . On the other hand,

$$|z_N(x)| \leq C_2 N x^{N(\alpha-2)} \exp\left(-\frac{N^2 \log^2 x}{4}\right).$$

If $|x - 1| \geq \delta$, then $\log^2 x \geq b > 0$, so that

$$\int_{1+\delta}^{\infty} |z_N(x)| x^{-1} dx \leq C_3 e^{-\frac{N^2 b}{4}} (1 + \delta)^{N(\alpha-2)} \rightarrow 0,$$

as $N \rightarrow \infty$, and (for $\delta < 1$)

$$\begin{aligned} \int_0^{1-\delta} |z_N(x)| x^{-1} dx &\leq C_2 N \int_{-\infty}^{\log(1-\delta)} \exp\left\{Nt(\alpha-2) - \frac{N^2 t^2}{4}\right\} dt \\ &= C_2 \int_{-\infty}^{N \log(1-\delta)} \exp\left\{s(\alpha-2) - \frac{s^2}{4}\right\} ds \rightarrow 0, \end{aligned}$$

as $N \rightarrow \infty$.

As a result, we see that, if N is large enough, the first summand in (16) does not exceed $2\|f\|_p \varepsilon$, while the second $\leq C_1 \varepsilon$. Thus,

$$\|z_N *_M f - f\|_p \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

■

2.3. On ratios of the Gamma functions. We will often use the function

$$\rho(t) = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad -1 < t < \infty. \quad (17)$$

Here we collect some of its properties.

If $t > \alpha - 1$, the integral representation

$$\frac{1}{\rho(t)} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-st} e^{(\alpha-1)s} (1 - e^{-s})^{\alpha-1} ds \quad (18)$$

holds (see Chapter 4 in [14]). It follows from (18) that the function $t \mapsto \frac{1}{\rho(t)}$ is strictly monotone decreasing and $\frac{1}{\rho(t)} \rightarrow 0$, as $t \rightarrow \infty$. Since $\Gamma(t+1-\alpha)$ has a pole at $t = \alpha - 1$, it is seen from (17) that $\rho(t) \rightarrow 0$, as $t \rightarrow \alpha - 1 + 0$.

If $-1 < t < \alpha - 1$, then (by a well-known identity for the Gamma function)

$$\rho(t) = \frac{\Gamma(-t + \alpha)}{\Gamma(-t)} \cdot \frac{\sin \pi(t + 1)}{\sin \pi(t + 1 - \alpha)}.$$

The integral representation for the ratio of the Gamma functions [14] leads, after an elementary investigation, to the conclusion that $\rho(t)$ is strictly monotone increasing from $-\infty$ to 0.

Thus, we conclude that on the interval $(-1, \infty)$ the function $\rho(t)$ is strictly monotone increasing from $-\infty$ to ∞ . The inverse function $\gamma(\lambda)$ solving the equation

$$\rho(t) = \lambda, \quad \lambda \in \mathbb{R},$$

is a well-defined continuous function. Note that $\rho(0) = \frac{1}{\Gamma(1 - \alpha)}$, so that

$$\gamma(\lambda) \geq 0, \text{ if } \lambda \geq \frac{1}{\Gamma(1 - \alpha)}.$$

It is known ([14], Chapter 4) that

$$\frac{\Gamma(t + a)}{\Gamma(t + b)} \sim t^{a-b}(1 + O(t^{-1})), \quad t \rightarrow \infty, \quad (19)$$

if $b > a$. In particular,

$$\rho(t) \sim t^\alpha, \quad \text{as } t \rightarrow \infty.$$

For the sequence $\beta(n)$ defined in (10) and appearing in the definition of the Gelfond-Leontiev generalized differentiation operator \mathfrak{D}_α , we have $\beta(n) = \rho(\alpha n)$, so that the above asymptotics implies the relation

$$\beta(n) \sim Cn^\alpha, \quad n = 0, 1, 2, \dots \quad (20)$$

3. α -Entire solutions

Let us consider α -entire solutions of the equation (2) with $a(t) = A(t^\alpha)$, where A is a polynomial of degree $m \geq 0$. We assume the initial condition $u(0) = u_0$.

THEOREM 1. *Under the above assumptions, the solution $u(t)$ of the equation (2) has the form $u(t) = v(t^\alpha)$, where v is an entire function whose order does not exceed $(1 + m)/\alpha$.*

P r o o f. Seeking the function v , we have $(\mathfrak{D}_\alpha v)(z) = a(z)v(z)$. Let us apply the operator \mathfrak{J}_α (see (11)) to both sides of this equality. We get from (11) and (12) that

$$v(Re^{i\theta}) - v(0) = \frac{e^{i\theta}}{\alpha\Gamma(\alpha)} \int_0^R \left[\left(\frac{R}{r} \right)^{1/\alpha} - 1 \right]^{\alpha-1} a(re^{i\theta}) v(re^{i\theta}) dr,$$

which implies the inequality

$$|v(Re^{i\theta})| \leq |v(0)| + C \int_0^R \left[\left(\frac{R}{r} \right)^{1/\alpha} - 1 \right]^{\alpha-1} |a(re^{i\theta})| |v(re^{i\theta})| dr$$

(here and below we denote by the same letter C various positive constants).

We have the asymptotic relations

$$t^{\frac{1}{\alpha}} - 1 \sim \frac{1}{\alpha}(t - 1), \quad \text{as } t \rightarrow 1 + 0;$$

$$\left(t^{\frac{1}{\alpha}} - 1 \right)^{\alpha-1} \sim t^{\frac{\alpha-1}{\alpha}}, \quad \text{as } t \rightarrow \infty.$$

Therefore

$$\left(t^{\frac{1}{\alpha}} - 1 \right)^{\alpha-1} \leq C(t - 1)^{\alpha-1} t^{2 - \frac{1}{\alpha} - \alpha}, \quad t \geq 1,$$

so that

$$|v(Re^{i\theta})| \leq |v(0)| + CR^{2 - \frac{1}{\alpha} - \alpha} \int_0^R (R - r)^{\alpha-1} r^{\frac{1}{\alpha} - 1} |a(re^{i\theta})| |v(re^{i\theta})| dr.$$

Since $a(z)$ is a polynomial of degree m , we get

$$|v(Re^{i\theta})| \leq |v(0)| + CR^{2 - \frac{1}{\alpha} - \alpha} \int_0^R (R - r)^{\alpha-1} r^{\frac{1}{\alpha} - 1 + m} |v(re^{i\theta})| dr,$$

where C does not depend on R, θ . Fixing θ and denoting

$$w(r) = \frac{|v(re^{i\theta})|}{r^{2 - \frac{1}{\alpha} - \alpha}},$$

we come to the inequality

$$w(R) \leq \frac{|v(0)|}{R^{2-\frac{1}{\alpha}-\alpha}} + C \int_0^R (R-r)^{\alpha-1} r^{1-\alpha+m} w(r) dr. \quad (21)$$

Now we are in a position to apply Henry's theorem (see Lemma 7.1.2 from [9]), which states that the inequality (21) implies the inequality

$$w(R) \leq \frac{|v(0)|}{R^{2-\frac{1}{\alpha}-\alpha}} \mathcal{E}_{\alpha, 2-\alpha+m}(CR)$$

where $\mathcal{E}_{\alpha, \sigma}(s)$ is a certain function admitting the estimate

$$\mathcal{E}_{\alpha, \sigma}(s) \leq C s^{\frac{1}{2}(\frac{\alpha+\sigma-1}{\alpha}-\sigma)} \exp\left(\frac{\alpha}{\alpha+\sigma-1} s^{\frac{\alpha+\sigma-1}{\alpha}}\right).$$

Thus,

$$|v(Re^{i\theta})| \leq C \exp\left(\mu R^{\frac{1+m}{\alpha}}\right)$$

for some $\mu \geq 0$, as desired. ■

4. Regular singularity

4.1. Formal and α -analytic solutions. Let us consider systems of equations of the form

$$t^\alpha (D_{0+}^\alpha u)(t) = A(t^\alpha)u(t), \quad (22)$$

where

$$A(z) = A_0 + \sum_{m=1}^{\infty} A_m z^m,$$

A_m are $n \times n$ complex matrices and

$$\|A_m\| \leq M \mu^m \quad (\mu > 0), \quad m = 0, 1, 2, \dots$$

Suppose we have a formal series

$$u(t) = \sum_{k=0}^{\infty} u_k t^{\alpha k}, \quad u_k \in \mathbb{C}^n. \quad (23)$$

Let us substitute the series (23) formally into (22). We get, in accordance with (6), that

$$\sum_{k=0}^{\infty} \beta(k) u_k t^{\alpha k} = \sum_{m,k=0}^{\infty} A_m u_k t^{\alpha(m+k)}$$

where $\beta(k)$ is the sequence (10). Collecting and comparing the terms we find that

$$\beta(l) u_l = \sum_{k=0}^l A_k u_{l-k}, \quad l = 0, 1, 2, \dots,$$

or, equivalently,

$$A_0 u_0 = \frac{1}{\Gamma(1-\alpha)} u_0; \quad (24)$$

$$[A_0 - \beta(l)] u_l = - \sum_{k=1}^l A_k u_{l-k}, \quad l \geq 1. \quad (25)$$

It is natural to call the formal series (23) a *formal solution* of the system (22) if the relations (24), (25) hold.

PROPOSITION 2. *If a formal series (23) is a formal solution of the system (22), then the series (23) is absolutely convergent on some neighbourhood of the origin.*

P r o o f. It follows from (19) that

$$\| [A_0 - \beta(l)]^{-1} \| \leq C l^{-\alpha}, \quad l \geq l_0.$$

In particular, we may assume that

$$\| [A_0 - \beta(l)]^{-1} \| \leq 1, \quad l \geq l_0.$$

Considering, if necessary, $\lambda u(t)$ instead of $u(t)$, with $|\lambda|$ small enough, we may assume that $\|u_0\| \leq 1$.

Let us choose so big $r > 0$ that $\|u_l\| \leq r^l$ for $l \leq l_0$ and

$$M \sum_{k=1}^{\infty} \left(\frac{\mu}{r} \right)^k \leq 1.$$

Then

$$\|u_l\| \leq r^l \quad \text{for all } l.$$

Indeed, if this inequality is proved up to some value of $l \geq l_0$, then

$$\|u_{l+1}\| \leq \left\| \sum_{k=1}^{l+1} A_k u_{l+1-k} \right\| \leq M \sum_{k=1}^{l+1} \mu^k r^{l+1-k} = M r^{l+1} \sum_{k=1}^{l+1} \left(\frac{\mu}{r}\right)^k \leq r^{l+1},$$

and the above inequality implying local convergence in (23) has been proved. \blacksquare

The above arguments remain valid for systems of the form

$$t^\alpha \left(\mathbb{D}^{(\alpha)} u \right) (t) = A(t^\alpha) u(t). \quad (26)$$

The only difference is that, instead of (24), we get the relation $A_0 u_0 = 0$, just as in the classical case (see [7]).

4.2. Model scalar equations. Consider the equation

$$t^\alpha \left(D_{0+}^\alpha \varphi \right) (t) = \lambda \varphi(t), \quad \lambda \in \mathbb{R}. \quad (27)$$

By the relation (6), a solution of the equation (27) is $\text{const} \cdot t^{\gamma(\lambda)}$, where γ (the inverse function to ρ) was defined in Section 2.3. For example, if $\lambda = 0$, then we have $\gamma(0) = \alpha - 1$.

If we consider an equation similar to (27), but with the Caputo-Dzhrbasyan derivative, that is

$$t^\alpha \left(\mathbb{D}^{(\alpha)} \varphi \right) (t) = \lambda \varphi(t), \quad (28)$$

then the constant function is a solution of (28) for $\lambda = 0$. Suppose that $\lambda \neq 0$, and φ is a solution of (28), that is $\varphi \in C[0, T]$, the function

$$(I_{0+}^{1-\alpha} \varphi) (t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \varphi(\tau) d\tau$$

is absolutely continuous, and (28) is satisfied with

$$\left(\mathbb{D}^{(\alpha)} \varphi \right) (t) = \frac{d}{dt} (I_{0+}^{1-\alpha} \varphi) (t) - \frac{1}{t^\alpha \Gamma(1-\alpha)} \varphi(0).$$

We have

$$(I_{0+}^{1-\alpha} \varphi) (t) = \frac{t^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} \varphi(st) ds,$$

and since φ is continuous,

$$(I_{0+}^{1-\alpha}\varphi)(t) \longrightarrow 0, \quad \text{as } t \rightarrow +0.$$

It is known [18] that in these circumstances $I_{0+}^\alpha D_{0+}^\alpha \varphi = \varphi$. Note also that I_{0+}^α transforms the function $t^{-\alpha}$ into the constant $\Gamma(1-\alpha)$. Dividing the equation (28) by t^α and applying I_{0+}^α to both sides, we find that

$$\begin{aligned} \varphi(t) - \varphi(0) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-1+\alpha} \tau^{-\alpha} \varphi(\tau) d\tau \\ &= \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{-1+\alpha} s^{-\alpha} \varphi(ts) ds. \end{aligned}$$

Passing to the limit, as $t \rightarrow 0$, and taking into account the continuity of φ , we obtain the identity

$$\frac{\lambda\varphi(0)}{\Gamma(\alpha)} \int_0^1 (1-s)^{-1+\alpha} s^{-\alpha} ds = 0,$$

whence $\varphi(0) = 0$.

Thus, for $\lambda \neq 0$, the equation (28) is equivalent to (27), if (27) is considered for continuous functions vanishing at the origin. The power solution $Ct^{\gamma(\lambda)}$ belongs to this class, if $\lambda > \frac{1}{\Gamma(1-\alpha)}$. It may be instructive to see these solutions, satisfying the equation

$$\varphi(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-1+\alpha} \tau^{-\alpha} \varphi(\tau) d\tau,$$

as examples of non-uniqueness of solutions of linear Volterra integral equations occurring due to the singularity of a kernel.

4.3. Systems with good spectrum. Let us consider the equation (3) with $A(z) = \sum_{m=0}^{\infty} A_m z^m$ where A_m are complex $n \times n$ matrices, the matrix A_0 is Hermitian, and the series converges on a neighbourhood of the origin. Without restricting generality, we may assume that

$$A_0 = \text{diag}(\lambda_1, \dots, \lambda_n), \quad \lambda_1, \dots, \lambda_n \in \mathbb{R}.$$

Following the classical method (see, for example, [1]) we look for a matrix-valued solution (*a fundamental solution*) of the equation (3), in the form

$$u(t) = S(t^\alpha)\psi(t) \quad (29)$$

where $\psi(t) = \text{diag}(t^{\gamma(\lambda_1)}, \dots, t^{\gamma(\lambda_n)})$, $S(z) = \sum_{\nu=0}^{\infty} \sigma_\nu z^\nu$, σ_ν ($\nu \geq 1$) are some unknown matrices, $\sigma_0 = I$.

We have

$$u(t) = \sum_{\nu=0}^{\infty} \sigma_\nu \text{diag}(t^{\gamma(\lambda_1)+\alpha\nu}, \dots, t^{\gamma(\lambda_n)+\alpha\nu}),$$

whence

$$t^\alpha (D_{0+}^\alpha u)(t) = \sum_{\nu=0}^{\infty} \sigma_\nu R_\nu \text{diag}(t^{\gamma(\lambda_1)+\alpha\nu}, \dots, t^{\gamma(\lambda_n)+\alpha\nu}), \quad (30)$$

where

$$R_\nu u = \text{diag}\left(\frac{\Gamma(\gamma(\lambda_1) + \alpha\nu + 1)}{\Gamma(\gamma(\lambda_1) + \alpha\nu + 1 - \alpha)}, \dots, \frac{\Gamma(\gamma(\lambda_n) + \alpha\nu + 1)}{\Gamma(\gamma(\lambda_n) + \alpha\nu + 1 - \alpha)}\right).$$

On the other hand,

$$A(t^\alpha)u(t) = \sum_{\nu=0}^{\infty} \left(\sum_{m=0}^{\nu} A_m \sigma_{\nu-m} \right) \text{diag}(t^{\gamma(\lambda_1)+\alpha\nu}, \dots, t^{\gamma(\lambda_n)+\alpha\nu}). \quad (31)$$

Note that $R_0 = \text{diag}(\lambda_1, \dots, \lambda_n) = A_0$, and since $\sigma_0 = I$, the coefficients corresponding to $\nu = 0$ in (30) and (31) coincide. Comparing the rest of the coefficients, we obtain the following system of equations for the matrices σ_k :

$$\sigma_k R_k - A_0 \sigma_k = \sum_{l=0}^{k-1} A_{k-l} \sigma_l, \quad k \geq 1. \quad (32)$$

For each k , the matrix equation (32) for σ_k has a unique solution if the spectra of the matrices R_k and A_0 are disjoint (see Appendix A.1 in [1]), that is

$$\frac{\Gamma(\gamma(\lambda_i) + \alpha k + 1)}{\Gamma(\gamma(\lambda_i) + \alpha k + 1 - \alpha)} \neq \lambda_j \quad (33)$$

for all $i, j \in \{1, \dots, n\}$, or, equivalently, since the left-hand side of (33) equals $\rho(\gamma(\lambda_i) + \alpha k)$,

$$\gamma(\lambda_j) - \gamma(\lambda_i) \neq \alpha k, \quad \text{for all } i, j \in \{1, \dots, n\}.$$

We call our system (3) *a system with good spectrum*, if

$$\gamma(\lambda_j) - \gamma(\lambda_i) \notin \alpha\mathbb{N}, \quad \text{for all } i, j \in \{1, \dots, n\}. \quad (34)$$

This definition extends the classical one [1], since for $\alpha = 1$ we would have $\rho(t) = \gamma(t) = t$, and the condition (34) would mean that the eigenvalues of A_0 must not differ by a natural number.

THEOREM 2. *If a system (3) has a good spectrum, then it possesses a fundamental solution (29) where the series for $S(z)$ has a positive radius of convergence.*

P r o o f. By the asymptotic relation (19),

$$\frac{\Gamma(\gamma(\lambda_j) + \alpha k + 1)}{\Gamma(\gamma(\lambda_j) + \alpha k + 1 - \alpha)} \sim (\alpha k)^\alpha (1 + O(k^{-1})), \quad k \rightarrow \infty,$$

for all $j = 1, \dots, n$. Therefore

$$(\alpha k)^{-\alpha} R_k = I + O(k^{-1}), \quad k \rightarrow \infty. \quad (35)$$

Let us divide both sides of the equation (32) by $(\alpha k)^\alpha$. The resulting equation, considered as a system of scalar equations for n^2 elements of the matrix σ_k , has the coefficients bounded in k and the determinant, which is different from zero for each k and tends to 1, as $k \rightarrow \infty$. This implies the estimate

$$\|\sigma_k\| \leq a k^{-\alpha} \left\| \sum_{l=0}^{k-1} A_{k-l} \sigma_l \right\|, \quad k \geq 1, \quad (36)$$

where the constant $a > 0$ does not depend on k . It follows from (35), (36), and the convergence near the origin of the power series for $A(z)$ that

$$\|\sigma_k\| \leq a_1 k^{-\alpha} \sum_{l=0}^{k-1} b^{k-l} \|\sigma_l\|, \quad k \geq 1,$$

where a_1 and b are positive constants independent of k .

Define a sequence $\{s_k\}_0^\infty$ of positive numbers, setting $s_0 = 1$,

$$s_k = a_1 k^{-\alpha} \sum_{l=0}^{k-1} b^{k-l} s_l, \quad k \geq 1.$$

The induction on k yields the inequality $\|\sigma_k\| \leq s_k$ for all $k \geq 0$. On the other hand,

$$\begin{aligned} s_{k+1} &= a_1(k+1)^{-\alpha} \sum_{l=0}^k b^{k+1-l} s_l \\ &= \frac{(k+1)^{-\alpha}}{k^{-\alpha}} \left[a_1 k^{-\alpha} \left(b \sum_{l=0}^{k-1} b^{k-l} s_l + b s_k \right) \right] \\ &= \frac{(k+1)^{-\alpha}}{k^{-\alpha}} (b s_k + a_1 k^{-\alpha} b s_k) = \frac{(k+1)^{-\alpha} b}{k^{-\alpha}} (1 + a_1 k^{-\alpha}) s_k. \end{aligned}$$

Therefore

$$\frac{s_k}{s_{k+1}} \longrightarrow b^{-1}, \quad \text{as } k \rightarrow \infty. \quad (37)$$

It follows from (37) that the series $\sum_{k=0}^{\infty} s_k z^k$ has the convergence radius b^{-1} (see Section 2.6 in [21]). Moreover, the series $\sum_{\nu=0}^{\infty} \sigma_{\nu} z^{\nu}$ converges for $|z| < b^{-1}$. ■

For the equation (26), a similar construction is valid, if we assume that $\lambda_1, \dots, \lambda_n \geq \frac{1}{\Gamma(1-\alpha)}$.

5. Irregular singularity: An example

5.1. A formal solution. In this section we construct a solution, in a sense to be specified, of the equation (5). Looking at classical first order equations, corresponding formally to $\alpha = 1$, we have to consider the equation $t^2 y'(t) = \lambda y(t)$ whose solution is $y(t) = \exp(-\lambda t^{-1})$. Therefore it is natural to seek a solution of the equation (5) in the form

$$u(t) = \sum_{n=0}^{\infty} c_n t^{-n\alpha}, \quad c_n \in \mathbb{C}. \quad (38)$$

A fractional derivative of any term in (38) with $n > \alpha^{-1}$ does not make sense classically. However we may apply the distribution theory from Section 2.2. Below we understand the fractional derivative $\mathbb{D}^{(\alpha)}$ in the sense of (13).

PROPOSITION 3. (i) If $\mu < 0$, $\mu \neq -1, -2, \dots$, then

$$\mathbb{D}^{(\alpha)} t^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \alpha)} t^{\mu - \alpha}. \quad (39)$$

(ii) If k is a natural number, then

$$\mathbb{D}^{(\alpha)} t^{-k} = \frac{(-1)^{k-1}}{(k-1)!\Gamma(-k+1-\alpha)} t^{-k-\alpha} \log t. \quad (40)$$

P r o o f. Let $\varphi \in \Phi_+^{1-\alpha}$. By (13),

$$\langle \mathbb{D}^{(\alpha)} t^\mu, \varphi(t) \rangle = \mu \langle t^{\mu-1}, (I_-^{1-\alpha} \varphi)(t) \rangle. \quad (41)$$

It is clear that the right-hand side of (41) is an entire function of μ . For $\mu > 0$, by virtue of (7),

$$\langle t^{\mu-1}, (I_-^{1-\alpha} \varphi)(t) \rangle = \langle I_{0+}^{1-\alpha} t^{\mu-1}, \varphi(t) \rangle = \frac{\Gamma(\mu)}{\Gamma(\mu+1-\alpha)} \langle t^{\mu-\alpha}, \varphi(t) \rangle.$$

For $\mu < 0$, $\mu \neq -1, -2, \dots$, the analytic continuation gives the equality (39).

Next, consider the entire function

$$F(\mu) = \langle t^{\mu-\alpha}, \varphi(t) \rangle, \quad \mu \in \mathbb{C}.$$

Note that $F(-k) = 0$, $k \in \mathbb{N}$, by the definition of the space $\Phi_+^{1-\alpha}$. We have

$$F'(\mu) = \langle \log t \cdot e^{(\mu-\alpha) \log t}, \varphi \rangle.$$

In particular,

$$F'(-k) = \int_0^\infty t^{-\alpha-k} \log t \cdot \varphi(t) dt.$$

As μ belongs to a small neighbourhood of the point $-k$, $F(\mu) = F'(-k)(\mu + k) + o(\mu + k)$. Since the residue of $\Gamma(\mu + 1)$ at $\mu = -k$ equals $\frac{(-1)^{k-1}}{(k-1)!}$ (see Section 4.4.1 of [20]), we see that the function $\Gamma(\mu + 1)F(\mu)$ is holomorphic at $\mu = -k$ (in fact, it is entire) and

$$\Gamma(\mu + 1)F(\mu)|_{\mu=-k} = \frac{(-1)^{k-1}}{(k-1)!} \int_0^\infty t^{-\alpha-k} \log t \cdot \varphi(t) dt.$$

Now the equality (39) implies (40). ■

Returning to (38), we will formally apply $\mathbb{D}^{(\alpha)}$ termwise and find the coefficients c_n comparing resulting terms in (5). It is clear from the equality (40) that such a procedure would fail if some of the numbers $n\alpha$ are integers. Thus, we have to assume that α is *irrational*. Using (39) we find that (formally)

$$t^{2\alpha} \left(\mathbb{D}^{(\alpha)} u \right) (t) = \sum_{n=1}^{\infty} c_n \frac{\Gamma(-n\alpha + 1)}{\Gamma(-n\alpha + 1 - \alpha)} t^{-n\alpha + \alpha}.$$

Substituting this into (5) we come to the recurrence relation

$$c_{n+1} = \lambda \frac{\Gamma(1 - (n+2)\alpha)}{\Gamma(1 - (n+1)\alpha)} c_n, \quad n \geq 0,$$

and it is easy to find by induction that

$$c_n = \lambda^n \frac{\Gamma(1 - (n+1)\alpha)}{\Gamma(1 - \alpha)} c_0.$$

Thus, we have found a formal solution

$$u(t) = \frac{c_0}{\Gamma(1 - \alpha)} \sum_{n=0}^{\infty} \Gamma(1 - (n+1)\alpha) t^{-n\alpha} \quad (42)$$

of the equation (5). Using the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

we can rewrite (42) in the form

$$u(t) = \frac{c_0 \pi}{\Gamma(1 - \alpha)} \sum_{n=0}^{\infty} \frac{1}{\sin(\pi(n+1)\alpha)} \frac{t^{-n\alpha}}{\Gamma((n+1)\alpha)}. \quad (43)$$

5.2. The convergence problem. The convergence of the series (43) depends on the arithmetic properties of the irrational number α . It was shown by Hardy [6] that α can be chosen in such a way (to be well approximated by rational numbers) that the series (43) would diverge for small values of t .

An irrational number $\alpha \in (0, 1)$ is said to be *poorly approximated by rational numbers*, if there exist such $\varepsilon > 0$, $c > 0$ that for any rational number $\frac{p}{q}$, $p, q \in \mathbb{N}$,

$$\left| \alpha - \frac{p}{q} \right| \geq cq^{-2-\varepsilon}. \quad (44)$$

By the Thue-Siegel-Roth theorem (see [19]) such are all algebraic numbers.

The first statement of the next theorem is actually contained already in the paper [6].

THEOREM 3. *If α is poorly approximated by rational numbers, then the series (43) converges for any $t > 0$. However this series diverges in the space of distributions $(\Phi_+^{1-\alpha})'$.*

P r o o f. It follows from (44) that

$$|q\alpha - p| \geq cq^{-1-\varepsilon} \quad \text{for any } p \in \mathbb{N},$$

so that

$$\text{dist}((n+1)\alpha, \mathbb{Z}_+) \geq c(n+1)^{-1-\varepsilon},$$

and taking $l \in \mathbb{Z}_+$, such that $|(n+1)\alpha - l| \leq \frac{1}{2}$, we find that

$$\begin{aligned} |\sin(\pi(n+1)\alpha)| &= |\sin(\pi((n+1)\alpha - l))| \geq 2|(n+1)\alpha - l| \\ &\geq 2 \text{dist}((n+1)\alpha, \mathbb{Z}_+) \geq 2c(n+1)^{-1-\varepsilon}. \end{aligned}$$

Using the Stirling formula we obtain that the series (43) converges for each $t > 0$.

To prove the second assertion, consider $\langle t^{-\alpha n}, \varkappa_\alpha(t) \rangle$, where the function \varkappa_α was defined in Section 2.2. We have, by (14), that

$$\begin{aligned} \langle t^{-\alpha n}, \varkappa_\alpha(t) \rangle &= \int_0^\infty t^{-\alpha n} \varkappa_\alpha(t) dt = \widetilde{\varkappa}_\alpha(1 - \alpha n) = \widetilde{\varkappa}(\alpha - 1 - \alpha n) \\ &= 2\sqrt{\pi} \exp\left((\alpha - 1 - \alpha n)^2 - \frac{\pi^2}{4}\right) \sin(\pi(\alpha - 1 - \alpha n)). \end{aligned}$$

Now we can give a lower estimate of the coefficients in the series (43) understood in the distribution sense: there are such $a, a_1, b > 0$ that

$$\begin{aligned}
& \left| \frac{1}{\sin(\pi(n+1)\alpha)} \cdot \frac{\langle t^{-n\alpha}, \varkappa_\alpha(t) \rangle}{\Gamma((n+1)\alpha)} \right| \\
& \geq a(n+1)^{-b(n+1)} |\sin(\pi\alpha(n-1))| \exp(\alpha^2 n^2 - 2\alpha(\alpha-1)n) \\
& \geq a_1(n+1)^{-b(n+1)-1-\varepsilon} \exp(\alpha^2 n^2 - 2\alpha(\alpha-1)n) \\
& = a_1 \exp \{ \alpha^2 n^2 - 2\alpha(\alpha-1)n - (bn + b + 1 + \varepsilon) \log(n+1) \} \rightarrow \infty,
\end{aligned}$$

as $n \rightarrow \infty$. Therefore the series (43) does not converge in $(\Phi_+^{1-\alpha})'$. ■

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